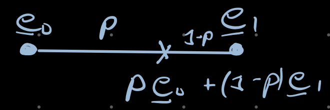
State space of a qubit

The state space of a classical system with 2 properties { "red", "blue" }, {0,1} i.e. a bit is a line segment



What about a qubit? $\rho = \rho^\dagger$, $Tr(\rho) = 1$, $\rho \geq 0$

Qubit = quantum system of dimension 2

$\Rightarrow$ 2×2 Hermitian matrix

$\rho = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$ to ensure $\rho = \rho^\dagger$

where a & b are real & c is complex

$Tr(\rho) = 1 \Rightarrow a + b = 1 \Rightarrow \rho = \underbrace{\begin{pmatrix} a & c \\ c^* & 1-a \end{pmatrix}}_{\text{3 degrees of freedom}}$

Any 2×2 Hermitian matrix can be written in the Pauli bases.

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

If you've not seen this before convince yourself of it. Hint: first show $Tr(\sigma_i \sigma_j) = 2\delta_{ij}$

Specifically, we have $\rho = a_0 (I + x\sigma_x + y\sigma_y + z\sigma_z)$ (for any Hermitian 2×2 matrix)

real numbers

Then $Tr(\rho) = 2a_0 = 1 \Rightarrow a_0 = 1/2$

The remaining condition to impose is $\text{eigs}(\rho) \geq 0$

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

Nice trick for eigenvals of 2x2 matrix
m = mean of diagonals
p = determinant
$\lambda = m \pm \sqrt{m^2 - p}$

$$\text{eigs}(\rho) = \frac{1}{2}\left(1 \pm \sqrt{\cancel{1} - (\cancel{1} - z^2 - (x^2+y^2))}\right)$$
$$= \frac{1}{2}\left(1 \pm |\underline{r}|\right) \geq 0 \implies \boxed{|\underline{r}| \leq 1}$$

$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Bloch vector must have norm ≤ 1

Or, conversely, any $\underline{r}$ specifies a unique quantum state

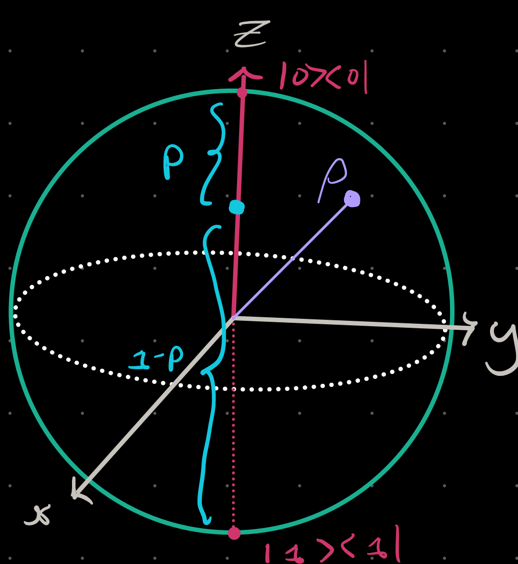
Therefore, the simplest state space in quantum theory is a ball of unit radius

for $\underline{r} = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}$ we get

$$\rho_\pm = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\rho_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (1\ 0)$$

$$\rho_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes (0\ 1)$$



& Classical theory is the line segment down the centre of the ball

$$\rho_p = p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$$
$$= \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$$

Ensembles of States

Given any two states σ_0 & σ_1 we can always form

$$\rho = p\,\sigma_a + (1-p)\,\sigma_b$$

for $0 \le p \le 1$

This is called an ensemble decomposition

operationally it can be thought of as preparing a state in σ_a with probability p

σ_b " " $1-p$

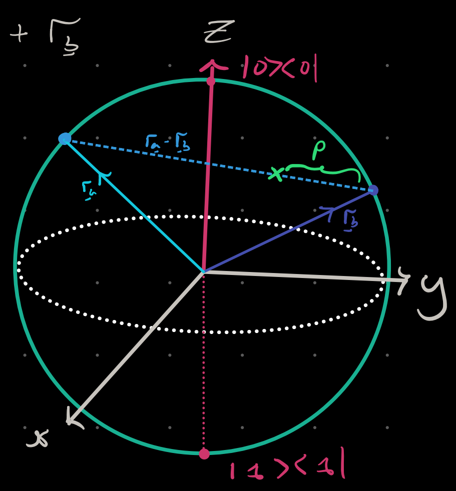
What does this look like in terms of Bloch vectors?

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$$

$$\sigma_a = \tfrac{1}{2}\left(I + \underline{r}_a \cdot \underline{\sigma} \right)$$

$$\sigma_b = \tfrac{1}{2}\left(I + \underline{r}_b \cdot \underline{\sigma} \right)$$

$$\rho = \tfrac{1}{2}\Big(I + \underbrace{\left(p\,\underline{r}_a + (1-p)\,\underline{r}_b\right)}_{p(\underline{r}_a - \underline{r}_b) + \underline{r}_b} \cdot \underline{\sigma} \Big)$$



Pure versus Mixed states

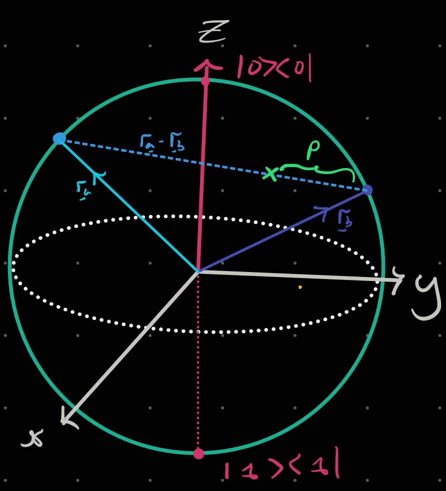
Of particular importance are <u>pure</u> states.

ρ is a pure state iff $\rho \neq p\,\sigma_0 + (1-p)\,\sigma_1$ for any two different states σ_0 & σ_1 for some probability $0 < p < 1$

i.e. pure states are any quantum states that cannot be obtained from mixing two states together

Any state that is not pure is called mixed.

Looking at the structure of the state space it's clear that we only have
$\rho \neq p\,\sigma_0 + (1-p)\,\sigma_1$
if the state is on the edge of the sphere with $|\underline{r}| = 1$



It is $\underbrace{\text{straightforward}}_{\text{(do it!)}}$ to show that in this case

$$\rho = \tfrac{1}{2}(I + \underset{\substack{\uparrow \\ |\underline{r}|}}{\underline{r}} \cdot \underline{\sigma}) \equiv |\psi\rangle\langle\psi| \quad \text{with} \quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

& so we are back to the pure states that most quantum courses start with...

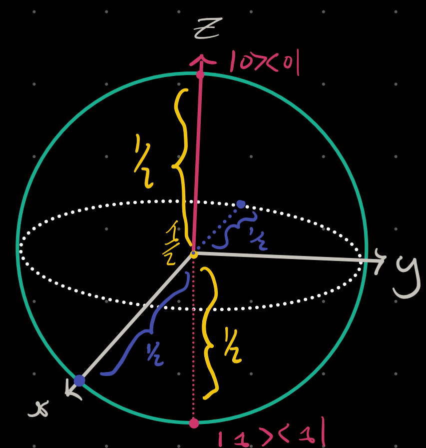
The ensemble ambiguity paradox

We saw that any classical state $\begin{pmatrix} p \\ 1-p \end{pmatrix}$ could be written as $\rho_z(p) = p\underbrace{|0\rangle\langle 0|}_{\begin{pmatrix}0\\0\\1\end{pmatrix}} + (1-p)\underbrace{|1\rangle\langle 1|}_{\begin{pmatrix}0\\0\\-1\end{pmatrix}}$

When $p = 1/2$ we have $\rho_z(p) = I/2$ } "Maximally mixed state"

$$r = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

But what if we took a mixture of $\rho_r^{\pm} = \frac{1}{2}(I \pm \underline{r}\cdot\underline{\sigma})$ for some $\underline{r}$



$$\rho_{\underline{r}}(p) = \frac{1}{2}\left(I + (2p-1)\,\underline{r}\cdot\underline{\sigma}\right)$$

$$\rho_{\underline{r}}(1/2) = I/2 = \rho_z(1/2)$$

Get the same state from 2 very different preparation procedures!

Is this weird? Kind of! For concreteness suppose $\underline{r} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

i.e. we have $\frac{I}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$

$\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

These two decompositions describe two very different preparation strategies.   0 = "Alive cat"   1 = "Dead cat"

① Fully classical: toss a coin - if heads - do nothing
if tails - kill cat

② Quantum! toss a coin - if heads - prepare Schrodinger cat state with + phase
if tails - prepare Schrödinger cat state with - phase.

But once prepared and given $\rho = I/2$ there is no measurement that can tell which preparation strategy was used!